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FINITE ELEMENT APPROXIMATION TO A CONTACT PROBLEM IN LINEAR THERMOELASTICITY

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ABSTRACT. A finite element approximation to the solution of a one-dimensional linear thermoelastic problem with unilateral contact of the Signorini type and heat flux is proposed. An error bound is derived and some numerical experiments are performed.

1. INTRODUCTION

We shall consider the numerical approximation of the following one-dimensional evolution problem with unilateral contact of the Signorini type:

$$D^2\tilde{\theta} = \tilde{\theta}_t + a(Du)_t \qquad x \in I,\ t > 0, \tag{1.1a}$$

$$D^2 u = aD\tilde{\theta} \qquad x \in I,\ t > 0, \tag{1.1b}$$

$$\tilde{\theta}(x,0) = p(x) \qquad x \in I, \tag{1.1c}$$

$$\tilde{\theta}(0,t) = \theta_A \qquad t > 0, \tag{1.1d}$$

$$-D\tilde{\theta}(1,t) = k\tilde{\theta}(1,t) \qquad t > 0, \tag{1.1e}$$

$$u(0,t) = 0 \qquad t > 0, \tag{1.1f}$$

$$u(1,t) \le g, \quad Du(1,t) \le a\tilde{\theta}(1,t) \qquad t > 0, \tag{1.1g}$$

$$(Du(1,t) - a\tilde{\theta}(1,t))(u(1,t) - g) = 0 \qquad t > 0, \tag{1.1h}$$

where $\tilde{\theta}(x,t)$ and $u(x,t)$ are the temperature and the displacement (parallel to the x-axis) of a homogeneous elastic body AB. The interval $I = (0,1)$ is the reference configuration of the body at the reference temperature $\tilde{\Theta}_r = 0$. The coupling constant a is usually small and is given in terms of physical parameters.

Here, $D \stackrel{def}{=} \frac{\partial}{\partial x}$, and $g > 0$ is a constant representing the gap between the end B $(x = 1)$ and an obstacle at temperature $\tilde{\theta} = 0$. At the end A $(x = 0)$, the body has a constant temperature θ_A and it is clamped. The right end $x = 1$ is free to expand or contract and the body may be in contact with the obstacle or not. However, the position of the right edge at $t > 0$ is not known a priori and the displacement cannot be more than g. A heat flux between the body and the obstacle is allowed with a constant heat transfer coefficient $k \ge 0$. The process is

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assumed to be slow and the acceleration term, present in the equations of linear thermoelasticity, has been omitted in (1.1b) leading to a quasi-static problem ([2]). We refer to Carlson ([3]) and Day ([6]) for physical background and mathematical modelling. Existence results to this problem were obtained by Andrews *et al.* in [1] using a heat transfer coefficient which may depend on the contact pressure and the distance of the right edge from the obstacle. Some numerical experiments using a finite difference scheme were performed by Shi *et al.* in [11].

Existence, uniqueness and regularity results to the solution of the contact problem with the temperature of the body being constant at both ends have been established by Copetti and Elliott in [4], Gilbert *et al.* in [9] and Shi and Shillor in [10]. In [4] Copetti and Elliott have also proposed and analysed a finite element approximation. The quasi-static problem with the body clamped at the boundary was studied by Day in [6]. The static case with unilateral Signorini boundary condition together with heat flux was investigated by Duvaut in [7] and an existence result to the full dynamic problem was obtained by Elliott and Qi in [8].

In this paper, we shall follow the work of Copetti and Elliott ([4]).

Following the idea introduced by Shi and Shillor in [10] we shall reformulate the problem into a decoupled form. Let us assume that the problem (1.1a)–(1.1h) has a classical solution. Integrating (1.1b) from x to 1 we obtain

$$Du(1,t) - Du(x,t) = a\left(\tilde{\theta}(1,t) - \tilde{\theta}(x,t)\right), \quad 0 \le x \le 1,\ t > 0, \tag{1.2}$$

and another integration with respect to x, from 0 to x, together with (1.1f) yields

$$xDu(1,t) - u(x,t) = a\int_0^x \left(\tilde{\theta}(1,t) - \tilde{\theta}(\xi,t)\right), d\xi \quad 0 \le x \le 1,\ t > 0, \tag{1.3}$$

and, in particular,

$$u(1,t) = Du(1,t) - a\int_0^1 \left(\tilde{\theta}(1,t) - \tilde{\theta}(x,t)\right) dx.$$

From (1.1h) we have

$$\left(u(1,t) - a\int_0^1 \tilde{\theta}(x,t)dx\right)(u(1,t) - g) = 0$$

and the condition $Du(1,t) \le a\tilde{\theta}(1,t)$ implies

$$u(1,t) \le a\int_0^1 \tilde{\theta}(x,t)dx.$$

Therefore

$$u(1,t) = \min\left\{a\int_0^1 \tilde{\theta}(x,t)dx, g\right\} \tag{1.4}$$

and

$$\begin{aligned} Du(1,t) &= \min\left\{a\int_0^1 \tilde{\theta}(x,t)dx, g\right\} + a\int_0^1 \left(\tilde{\theta}(1,t) - \tilde{\theta}(x,t)\right) dx \\ &= -\max\left\{a\int_0^1 \tilde{\theta}(x,t)dx - g, 0\right\} + a\tilde{\theta}(1,t). \end{aligned} \tag{1.5}$$

Using this result in equation (1.2) gives

$$Du(x,t) = -\max\left\{a\int_0^1 \tilde{\theta}(x,t)dx - g, 0\right\} + a\tilde{\theta}(x,t). \tag{1.6}$$

Differentiating this equation with respect to t we obtain

$$(Du)_t(x,t) = a\tilde{\theta}_t(x,t) - \frac{d}{dt}\max\left\{a\int_0^1 \tilde{\theta}(x,t)dx - g, 0\right\}$$

and equation (1.1a) may be written as

$$(1+a^2)\tilde{\theta}_t = D^2\tilde{\theta} + a^2\frac{d}{dt}\max\left\{\int_0^1 \tilde{\theta}(x,t)dx - \frac{g}{a}, 0\right\}.$$

Recalling (1.3) and (1.5) we find that the displacement is given by

$$u(x,t) = -ax\max\left\{\int_0^1 \tilde{\theta}(x,t)dx - \frac{g}{a}, 0\right\} + a\int_0^x \tilde{\theta}(\xi,t)d\xi.$$

The following existence and uniqueness result was obtained by Andrews *et al.* in [1]:

Theorem 1.1. *Given $p(x) \in H^1(I)$ with $p(0) = \theta_A$ and $k \in C^1(\mathbf{R})$, $k \geq 0$, there exists a unique $\tilde{\theta} \in H^{2,1}(\Omega_T)$ satisfying*

$$\begin{aligned}
&(1+a^2)\tilde{\theta}_t = D^2\tilde{\theta} + a^2\tfrac{d}{dt}\max\left\{\textstyle\int_0^1 \tilde{\theta}(x,t)dx - \frac{g}{a}, 0\right\}, && \text{a.e. in } \Omega_T,\\
&\tilde{\theta}(x,0) = p(x), && x \in I,\\
&\tilde{\theta}(0,t) = \theta_A, && 0 < t < T,\\
&-D\tilde{\theta}(1,t) = k\tilde{\theta}(1,t), && \text{a.e. in } (0,T),
\end{aligned}$$

where $\Omega_T = I \times (0,T)$, provided $0 < a < 1$.

Remark 1.2. In the work of Andrews *et al.*, in [1], $\theta_A = 1$ and the heat transfer coefficient $k(\cdot)$ is a function of $g - u(1,t) + \tilde{\sigma}(t)$, where $\tilde{\sigma}(t) = Du(1,t) - a\tilde{\theta}(1,t)$.

Remark 1.3. It would be natural to consider the general case where the temperature at $x = 0$ is time dependent and may be different from the value of the initial temperature at $x = 0$. This was done by Copetti and Elliott in [4] when the temperature of the body is constant at both ends.

Letting $\theta(x,t) = \tilde{\theta}(x,t) + \theta_A(x-1)$ we obtain the contact problem with homogeneous boundary condition at $x = 0$ for the temperature

$$(1+a^2)\theta_t - D^2\theta = a^2\frac{d}{dt}[\gamma]_+, \quad x \in I,\ t > 0, \tag{1.7a}$$

$$\gamma(t) := (1, \theta(t)) - r, \quad t > 0, \tag{1.7b}$$

$$\theta(x,0) = \theta_0(x) = p(x) + \theta_A(x-1), \quad x \in I, \tag{1.7c}$$

$$-D\theta(1,t) = k\theta(1,t) - \theta_A, \quad t > 0, \tag{1.7d}$$

$$\theta(0,t) = 0, \quad t > 0, \tag{1.7e}$$

where $(\cdot,\cdot)$ is the L^2-inner product, $[\gamma(t)]_+ := \max\{\gamma(t), 0\}$ and $r = \frac{g}{a} - \frac{\theta_A}{2}$. Note that $\tilde{\sigma}(t) = -a[\gamma(t)]_+$.

Throughout this paper, we denote the norms of $L^2(I)$ and $H^s(I)$ by $\|\cdot\|$ and $\|\cdot\|_s$, respectively. The semi-norm $\|Dv\|$ is indicated by $|v|_1$.

2. The finite element approximation of $\{\theta,\ \gamma\}$

Integrating (1.7a) against test functions χ in $H^1_E(I) = \{v \in H^1(I)|v(0) = 0\}$ and using the boundary conditions we obtain

$$(1 + a^2)(\theta_t, \chi) + (D\theta, D\chi) + k\theta(1,t)\chi(1) = \theta_A\chi(1) + a^2\frac{d}{dt}[\gamma]_+(1,\chi). \tag{2.1}$$

Let $0 = x_0 < x_1 < \ldots < x_s = 1$ be an equidistant partition of the interval $(0,1)$ into subintervals $I_j = (x_{j-1}, x_j)$, $j = 1, \ldots, s$, of length $h = \frac{1}{s}$ and denote by S^h_E the finite element space

$$S^h_E = \left\{\chi \in C^0(\overline{I}) : \chi|_{I_j} \in P_1,\ \chi(0) = 0\right\},$$

where P_1 is the space of linear functions.

The discrete Galerkin method for (1.7a)–(1.7e) is to find $\Theta^n \in S^h_E$ and Γ^n, $n = 1, \ldots, N$, such that $\forall\chi \in S^h_E$

$$(1 + a^2)\left(\frac{\Theta^n - \Theta^{n-1}}{\Delta t}, \chi\right) + (D\Theta^n, D\chi) + k\Theta^n(1)\chi(1)$$
$$= \theta_A\chi(1) + a^2\left(\frac{[\Gamma^n]_+ - [\Gamma^{n-1}]_+}{\Delta t}, \chi\right), \tag{2.2a}$$

$$\Gamma^n = (1, \Theta^n) - r, \tag{2.2b}$$

with Θ^0 the L^2-projection of θ_0 and $\Gamma^0 = \gamma(0)$ and where $\Delta t = \frac{T}{N}$. This choice of Θ^0 turns out to be convenient for the error analysis.

Given Θ^{n-1} we iterate to find Θ^n :

$$(1 + a^2)(\Theta^n_l, \chi) + \Delta t(D\Theta^n_l, D\chi) + \Delta tk\Theta^n_l(1)\chi(1)$$
$$= (1 + a^2)(\Theta^{n-1}, \chi) + \Delta t\theta_A\chi(1) + a^2([\Gamma^n_{l-1}]_+ - [\Gamma^{n-1}]_+, \chi), \tag{2.3}$$

where $\Gamma^n_{l-1} = (1, \Theta^n_{l-1}) - r$.

Theorem 2.1. *There exists a unique sequence* $\{\Theta^n\}^N_{n=1}$ *solving* (2.2a) *and* (2.2b).

Proof. Writing

$$\Theta^n_l = \sum_{i=1}^{s} c^{n+1}_{l,i}\chi_i,$$

where $\{\chi_i\}^s_{i=1}$ is the piecewise linear basis for S^h_E, and taking $\chi = \chi_j,\ j = 1, \ldots, s$, in (2.3) results in

$$(1 + a^2)\sum_{i=1}^{s} c^n_{l,i}(\chi_i, \chi_j) + \Delta t\sum_{i=1}^{s} c^n_{l,i}(D\chi_i, D\chi_j) + \Delta tk\sum_{i=1}^{s} c^n_{l,i}\chi_i(1)\chi_j(1)$$
$$= (1 + a^2)\sum_{i=1}^{s} c^{n-1}_i(\chi_i, \chi_j) + \Delta t\theta_A\chi_j(1) + a^2([\Gamma^n_{l-1}]_+ - [\Gamma^{n-1}]_+, \chi_j).$$

Hence, given Θ^n_{l-1}, we have to solve the system

$$\left((1 + a^2)M + \Delta tK + \Delta tkB\right)\underline{c}^n_l = (1 + a^2)M\underline{c}^{n-1} + \Delta t\theta_A\underline{d} + a^2\underline{e},$$

where

$$M_{ij} = (\chi_i, \chi_j), \quad K_{ij} = (D\chi_i, D\chi_j),$$
$$\underline{c}_l^n = \{c_{l,i}^n\}, \ \underline{c}^{n-1} = \{c_i^{n-1}\},$$
$$B_{ij} = 0 \ i, j \neq s, \ B_{ss} = 1,$$
$$\{\underline{d}\}_i = 0 \ i \neq s, \ \{\underline{d}\}_s = 1,$$
$$\{\underline{e}\}_i = ([\Gamma_{l-1}^n]_+ - [\Gamma^{n-1}]_+)h \ i \neq s, \ \{\underline{e}\}_s = ([\Gamma_{l-1}^n]_+ - [\Gamma^{n-1}]_+)\frac{h}{2}.$$

Since $k \geq 0$, $(1+a^2)M + \Delta t K + \Delta t k B$ is invertible and the above system has a unique solution $\underline{c}_l^n$.

Defining $\mathcal{F} : S_E^h \to S_E^h$ such that, for $\Theta \in S_E^h$, $\mathcal{F}(\Theta)$ satisfies, for any $\chi \in S_E^h$,

$$(1+a^2)(\mathcal{F}(\Theta), \chi) + \Delta t(D\mathcal{F}(\Theta), D\chi) + \Delta t k \mathcal{F}(\Theta)(1)\chi(1)$$
$$= (1+a^2)(\Theta^{n-1}, \chi) + \Delta t \theta_A \chi(1) + a^2([\Gamma(\Theta)]_+ - [\Gamma^{n-1}]_+, \chi), \tag{2.4}$$

where $\Gamma(\Theta) = (1, \Theta) - r$, it follows that $\mathcal{F}$ is well defined and (2.2a) and (2.2b) has a unique solution if $\mathcal{F}$ has a unique fixed point.

Let us take Θ, $\eta \in S_E^h$. By (2.4), $\forall \chi \in S_E^h$,

$$(1+a^2)(\mathcal{F}(\Theta) - \mathcal{F}(\eta), \chi) + \Delta t(D(\mathcal{F}(\Theta) - \mathcal{F}(\eta)), D\chi)$$
$$+\Delta t k \left(\mathcal{F}(\Theta)(1) - \mathcal{F}(\eta)(1)\right)\chi(1) = a^2([\Gamma(\Theta)]_+ - [\Gamma(\eta)]_+, \chi).$$

Choosing $\chi = \mathcal{F}(\Theta) - \mathcal{F}(\eta)$, we obtain

$$(1+a^2)\|\mathcal{F}(\Theta) - \mathcal{F}(\eta)\|^2 + \Delta t|\mathcal{F}(\Theta) - \mathcal{F}(\eta)|_1^2 + \Delta t k[\mathcal{F}(\Theta)(1) - \mathcal{F}(\eta)(1)]^2$$
$$\begin{aligned} &\leq a^2\|[\Gamma(\Theta)]_+ - [\Gamma(\eta)]_+\| \ \|\mathcal{F}(\Theta) - \mathcal{F}(\eta)\| \\ &\leq a^2\|\Theta - \eta\| \ \|\mathcal{F}(\Theta) - \mathcal{F}(\eta)\| \\ &\leq \frac{a^2}{2}\|\Theta - \eta\|^2 + \frac{a^2}{2}\|\mathcal{F}(\Theta) - \mathcal{F}(\eta)\|^2. \end{aligned}$$

Using $k \geq 0$ results in

$$\left(1 + \frac{a^2}{2}\right)\|\mathcal{F}(\Theta) - \mathcal{F}(\eta)\|^2 \leq \frac{a^2}{2}\|\Theta - \eta\|^2.$$

Therefore,

$$\|\mathcal{F}(\Theta) - \mathcal{F}(\eta)\|^2 \leq \frac{a^2}{2+a^2}\|\Theta - \eta\|^2$$

and $\mathcal{F}$ is a contraction on S_E^h. Thus, the sequence $\{\Theta_l^n\}$ defined by (2.3) converges to the unique solution of (2.2a) and (2.2b) for any choice of Θ_0^n. □

Next, we derive an error bound for the approximation (2.2a) and (2.2b). We will use the projection $P_E^h : H_E^1 \to S_E^h$ defined by

$$(DP_E^h v, D\chi) = (Dv, D\chi) \quad \forall \chi \in S_E^h. \tag{2.5}$$

By [5] we have

$$P_E^h v(x_i) = v(x_i) \ i = 0, 1, \ldots, s, \tag{2.6a}$$

$$\|v - P_E^h v\| + h\|Dv - DP_E^h v\| \leq Ch^2\|v\|_2. \tag{2.6b}$$

Theorem 2.2. *Let $\{\theta(t), \gamma(t)\}$ be the solution of* (1.7a)–(1.7e) *and $\{\Theta^n, \Gamma^n\}$ be the solution of* (2.2a) *and* (2.2b). *Then*

$$\Delta t \sum_{n=1}^{N} \|\Theta^n - \theta(t_n)\|^2 \leq C(\theta, T)(h^4 + (\Delta t)^2)$$

and, as a consequence,

$$\|\Theta^n - \theta(t_n)\| \leq C(\theta, T)\left(\frac{h^2}{(\Delta t)^{1/2}} + (\Delta t)^{1/2}\right),$$

$$|\Gamma^n - \gamma(t_n)| \leq C(\theta, T)\left(\frac{h^2}{(\Delta t)^{1/2}} + (\Delta t)^{1/2}\right),$$

with C a constant independent of h and Δt and $t_n = n\Delta t$.

Proof. Observe that we need only derive the first estimate. Let $\theta^n \equiv \theta(t_n)$ and $\gamma^n \equiv \gamma(t_n)$. Integrating equation (2.1) from 0 to t_n and summing (2.2a) from 1 to n, we obtain $\forall \chi \in S_E^h$

$$(1 + a^2)(\theta^n, \chi) - (1 + a^2)(\theta_0, \chi) + (D\hat{\theta}^n, D\chi) + k\chi(1)\left(\int_0^{t_n} \theta(1, t)dt\right)$$

$$= t_n\theta_A\chi(1) + a^2([\gamma^n]_+ - [\gamma(0)]_+)(1, \chi), \tag{2.7}$$

$$(1 + a^2)(\Theta^n, \chi) - (1 + a^2)(\Theta^0, \chi) + \left(\Delta t \sum_{i=1}^{n} D\Theta^i, D\chi\right) + \Delta t k\chi(1) \sum_{i=1}^{n} \Theta^i(1)$$

$$= t_n\theta_A\chi(1) + a^2([\Gamma^n]_+ - [\Gamma^0]_+)(1, \chi), \tag{2.8}$$

where $\hat{\theta}^n = \int_0^{t_n} \theta(s)ds$. Thus, using (2.5), we have $\forall \chi \in S_E^h$

$$(1 + a^2)(\Theta^n - \theta^n, \chi) - \left(D\left(P_E^h\hat{\theta}^n - \Delta t \sum_{i=1}^{n} \Theta^i\right), D\chi\right)$$

$$+k\chi(1)\left(\Delta t \sum_{i=1}^{n} \Theta^i(1) - \hat{\theta}^n(1)\right) = a^2([\Gamma^n]_+ - [\gamma^n]_+, \chi). \tag{2.9}$$

Let us define

$$\begin{aligned}
\varepsilon^j &= \Delta t \sum_{i=1}^{j} \Theta^i - P_E^h\hat{\theta}^j, \quad j = 1, \ldots, n, \\
\varepsilon^0 &= 0, \\
\overline{\theta}^j &= \frac{1}{\Delta t}\int_{t_{j-1}}^{t_j} \theta(s)ds, \quad j = 1, \ldots, n.
\end{aligned}$$

It follows that

$$\frac{\varepsilon^j - \varepsilon^{j-1}}{\Delta t} = \Theta^j - P_E^h\overline{\theta}^j$$

and by (2.9), $\forall \chi \in S_E^h$

$$(1 + a^2)\left(\frac{\varepsilon^n - \varepsilon^{n-1}}{\Delta t}, \chi\right) + (D\varepsilon^n, D\chi) + k\chi(1)\left(\Delta t \sum_{i=1}^{n} \Theta^i(1) - \hat{\theta}^n(1)\right) = I_1 + I_2,$$

where

$$I_1 = (1+a^2)(\theta^n - P_E^h \overline{\theta}^n, \chi)$$

and

$$I_2 = a^2([\Gamma^n]_+ - [\gamma^n]_+, \chi).$$

Thus

$$(1+a^2)\Big(\frac{\varepsilon^n - \varepsilon^{n-1}}{\Delta t}, \chi\Big) + (D\varepsilon^n, D\chi) + k\chi(1)\Big(\varepsilon^n(1) + P_E^h\hat{\theta}^n(1) - \hat{\theta}^n(1)\Big) = I_1 + I_2$$

and (2.6a) implies that

$$(2.10) \qquad (1+a^2)\left(\frac{\varepsilon^n - \varepsilon^{n-1}}{\Delta t}, \chi\right) + (D\varepsilon^n, D\chi) + k\chi(1)\varepsilon^n(1) = I_1 + I_2.$$

By applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} I_1 &= (1+a^2)(\theta^n - \overline{\theta}^n + \overline{\theta}^n - P_E^h\overline{\theta}^n, \chi) \\ &= (1+a^2)\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(s - t_{n-1})\theta_t(s)ds + \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(\theta(s) - P_E^h\theta(s))ds, \chi\right) \\ &\leq (1+a^2)\|\chi\|A_{n-1}, \end{aligned}$$

where

$$A_{n-1} = \frac{1}{\Delta t}\left(\left(\frac{(\Delta t)^3}{3}\int_{t_{n-1}}^{t_n}\|\theta_t(s)\|^2 ds\right)^{\frac{1}{2}} + \left(\Delta t\int_{t_{n-1}}^{t_n}\|\ \theta(s) - P_E^h\theta(s)\|^2 ds\right)^{\frac{1}{2}}\right),$$

and

$$\begin{aligned} I_2 &\leq a^2\|\chi\|\|\Theta^n - \theta^n\| \\ &\leq a^2\|\chi\|\|\Theta^n - P_E^h\overline{\theta}^n + P_E^h\overline{\theta}^n - \theta^n\| \\ &\leq A_{n-1}a^2\|\chi\| + a^2\|\chi\|\left\|\frac{\varepsilon^n - \varepsilon^{n-1}}{\Delta t}\right\|. \end{aligned}$$

Taking $\chi = \frac{\varepsilon^n - \varepsilon^{n-1}}{\Delta t}$ in (2.10) results in

$$\frac{1+a^2}{(\Delta t)^2}\|\varepsilon^n - \varepsilon^{n-1}\|^2 + \frac{1}{\Delta t}(D\varepsilon^n, D(\varepsilon^n - \varepsilon^{n-1})) + \frac{k}{\Delta t}(\varepsilon^n(1) - \varepsilon^{n-1}(1))\varepsilon^n(1)$$

$$\leq \frac{a^2}{(\Delta t)^2}\|\varepsilon^n - \varepsilon^{n-1}\|^2 + \frac{C}{\Delta t}\|\varepsilon^n - \varepsilon^{n-1}\|A_{n-1}.$$

Therefore,

$$\begin{aligned} &\frac{1}{(\Delta t)^2}\|\varepsilon^n - \varepsilon^{n-1}\|^2 + \frac{1}{2\Delta t}\left(|\varepsilon^n - \varepsilon^{n-1}|_1^2 + |\varepsilon^n|_1^2 - |\varepsilon^{n-1}|_1^2\right) \\ &\quad + \frac{k}{2\Delta t}(\varepsilon^n(1) - \varepsilon^{n-1}(1))^2 + \frac{k}{2\Delta t}\left((\varepsilon^n(1))^2 - (\varepsilon^{n-1}(1))^2\right) \\ &\leq \frac{1}{2(\Delta t)^2}\|\varepsilon^n - \varepsilon^{n-1}\|^2 + CA_{n-1}^2 \end{aligned}$$

and

$$\frac{1}{(\Delta t)^2}\sum_{j=1}^{n}\|\varepsilon^j-\varepsilon^{j-1}\|^2+\frac{1}{\Delta t}\sum_{j=1}^{n}|\varepsilon^j-\varepsilon^{j-1}|_1^2+\frac{1}{\Delta t}|\varepsilon^n|_1^2$$
$$+\frac{k}{\Delta t}\sum_{j=1}^{n}(\varepsilon^j(1)-\varepsilon^{j-1}(1))^2+\frac{k}{\Delta t}(\varepsilon^n(1))^2$$
$$\leq\frac{1}{\Delta t}|\varepsilon^0|_1^2+\frac{k}{\Delta t}(\varepsilon^0(1))^2+C\sum_{j=1}^{n}A_{j-1}^2.$$

Since $k\geq 0$, $\varepsilon^0=0$ and (2.6b) holds, the regularity result given by Theorem 1.1 and the definition of A_{j-1} yields

$$\Delta t\sum_{j=1}^{n}\left\|\frac{\varepsilon^j-\varepsilon^{j-1}}{\Delta t}\right\|^2+|\varepsilon^n|_1^2+k(\varepsilon^n(1))^2\leq C(\theta,T)(h^4+(\Delta t)^2).$$

Noting that

$$\Theta^n-\theta(t_n)=\Theta^n-P_E^h\overline{\theta}^n+P_E^h\overline{\theta}^n-\theta(t_n)$$

the result follows. □

3. The finite element approximation of u

A natural approximation to

$$u(x,t_n)=-ax[\gamma(t_n)]_++a\int_0^x\theta(\xi,t_n)d\xi+a\theta_A\left(x-\frac{x^2}{2}\right) \tag{3.1}$$

is given by $U^n\in K^h$, where $K^h=\{v\in C^0(\overline{I}):v|_{I_j}\in P_1, v(0)=0,\ v(1)\leq g\}$, defined by

$$U^n(x_j)=-ax_j[\Gamma^n]_++a\int_0^{x_j}\Theta^n(x)dx+a\theta_A\left(x_j-\frac{x_j^2}{2}\right),\quad j=1,\dots,s, \tag{3.2}$$

satisfying the following error bound.

Theorem 3.1. *Suppose that $u(t_n)$ is given by (3.1). Then*

$$|u(x_j,t_n)-U^n(x_j)|\leq C(\theta,T)\left(\frac{h^2}{(\Delta t)^{1/2}}+(\Delta t)^{1/2}\right).$$

Proof. Observe that

$$u(x_j,t_n)-U^n(x_j)=ax_j([\Gamma^n]_+-[\gamma^n]_+)+a\int_0^{x_j}(\theta^n(x)-\Theta^n(x))dx$$

and so

$$\begin{aligned}|u(x_j,t_n)-U^n(x_j)| &\leq a|[\Gamma^n]_+-[\gamma^n]_+|+a\|\theta^n-\Theta^n\|\\ &\leq 2a\|\theta^n-\Theta^n\|.\end{aligned}$$

The result is now a consequence of the error bound for the approximation of the temperature. □

4. The steady-state

The stationary problem in (1.1a)–(1.1h) is to find $\{\hat{\theta}(x), \hat{u}(x)\}$ such that

$$
\begin{aligned}
&D^2\hat{\theta} = 0, && x \in I,\\
&D^2\hat{u} = aD\hat{\theta}, && x \in I,\\
&\hat{\theta}(0) = \theta_A, &&\\
&-D\hat{\theta}(1) = k\hat{\theta}(1), &&\\
&\hat{u}(0) = 0, &&\\
&\hat{u}(1) \le g, \quad D\hat{u}(1) \le a\hat{\theta}(1), &&\\
&(D\hat{u}(1) - a\hat{\theta}(1))(\hat{u}(1) - g) = 0. &&
\end{aligned}
$$

We can see that $\{\hat{\theta}, \hat{u}\}$ given by

$$\hat{\theta}(x) = \theta_A + (\hat{\theta}(1) - \theta_A)x, \tag{4.1}$$

$$\hat{u}(x) = a\theta_A x + \frac{a}{2}(\hat{\theta}(1) - \theta_A)x^2 \qquad \text{if } \frac{a}{2}(\hat{\theta}(1) + \theta_A) < g, \tag{4.2a}$$

$$\hat{u}(x) = (\hat{\sigma} + a\theta_A)x + \frac{a}{2}(\hat{\theta}(1) - \theta_A)x^2 \qquad \text{if } \frac{a}{2}(\hat{\theta}(1) + \theta_A) > g, \tag{4.2b}$$

where $\hat{\sigma} = D\hat{u}(1) - a\hat{\theta}(1)$, is the solution to this problem, and

$$
\begin{aligned}
\hat{\theta}(1) &= \frac{\theta_A}{1+k},\\
\hat{u}(1) &= \min\left\{\frac{a}{2}(\hat{\theta}(1) + \theta_A),\ g\right\}.
\end{aligned}
$$

Hence, for fixed a, k and g, there is contact or not depending on θ_A. Taking $\theta_A > 0$, it follows that $0 < \hat{\theta}(1) \le \theta_A$ and therefore, at the steady-state, contact will not be observed for $g > a\theta_A$. If $g < \frac{a}{2}\theta_A$ there will be contact with the obstacle.

When $k = 0$ we have $\hat{\theta}(1) = \theta_A$ and $\hat{u}(1) = \min\{a\theta_A, g\}$. In the limit case, $k \to +\infty$, we find that

$$
\begin{aligned}
&\hat{\theta}(x) \to \theta_A(1 - x),\\
&\hat{\theta}(1) \to 0,\\
&-D\hat{\theta}(1) \to \theta_A,\\
&\hat{u}(1) \to \min\{\tfrac{a}{2}\theta_A, g\},\\
&\hat{\sigma} \to \min\{g - \tfrac{a}{2}\theta_A, 0\},
\end{aligned}
$$

with no temperature difference between the end B and the obstacle.

5. Numerical experiments

In our numerical simulations we took $\Delta t = h^2$, $a = 0.017$ and $g = 0.1$; the value for a was taken from the work by Gilbert *et al.*, in [9]. As an initial guess to Θ^n we choose Θ^{n-1} and the iterative process (2.3) was stopped when the difference between successive iterates was less than or equal to 1.0×10^{-7}. We let $h = \frac{1}{101}$, $p(x) = \theta_A \cos 2\pi x$ and Θ^0 was the interpolant of θ_0. Numerical integration, namely the trapezoidal rule, was used to compute M with the resulting matrix M being diagonal with diagonal elements $M_{ii} = h$, $i \neq s$, $M_{ss} = \frac{h}{2}$. Note that the temperature $\tilde{\theta}$ is shown in the pictures.

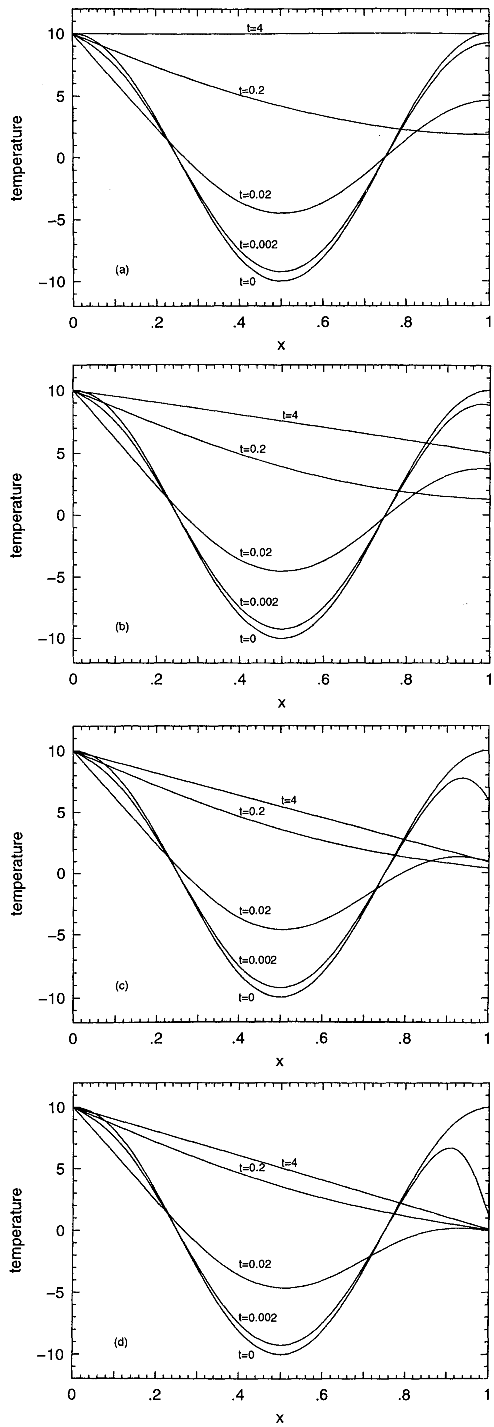

FIGURE 1. The evolution in time of the temperature from the initial condition when $\theta_A = 10$ for (a) $k = 0$, (b) $k = 1$, (c) $k = 10$ and (d) $k = 100$.

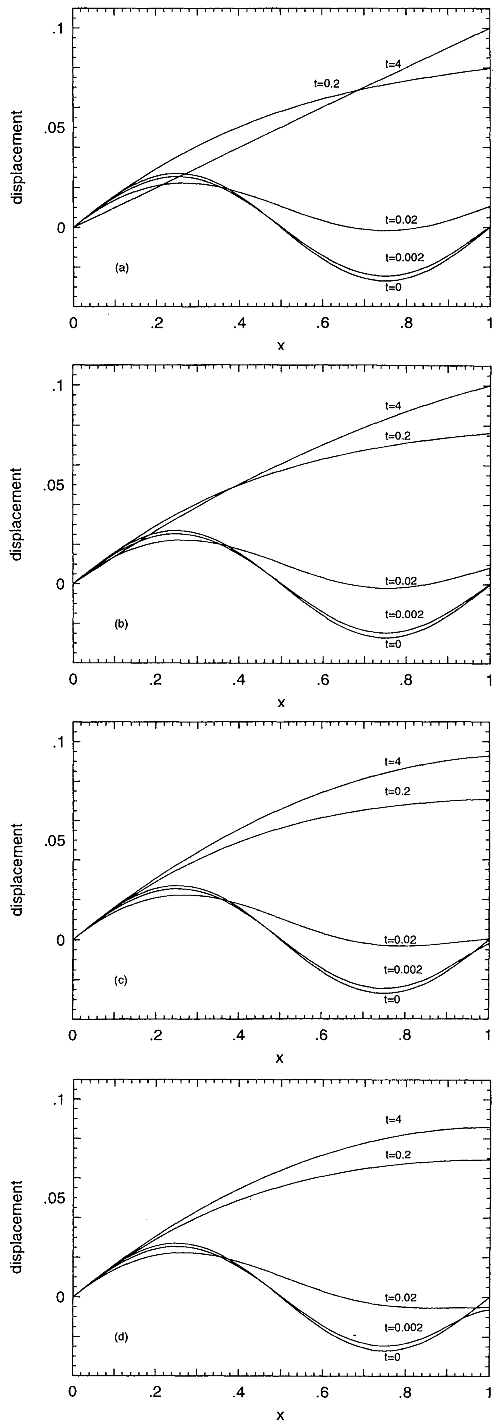

FIGURE 2. The evolution in time of the displacement when $\theta_A = 10$ for (a) $k = 0$, (b) $k = 1$, (c) $k = 10$ and (d) $k = 100$.

To investigate the convergence to the steady-state solution and the contact condition we performed four experiments. The numerical results are presented in Figures 1 and 2 where the temperature and the displacement are shown for $t = 0, 0.002$, 0.02, 0.2 and 4. The graphs did not change after the final state shown and the computations were stopped.

We fixed $\theta_A = 10$ and took increasing values of the heat transfer coefficient k, $k = 0$, 1, 10 and 100. Contact is observed when $k = 0$ and $k = 1$ and for $k = 100$, $\tilde{\theta}(1)$ is very small.

In all simulations, the numerical results are in agreement with the theoretical results given above.

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